# Public-Key Cryptography

Number Theory, RSA, Diffie-Hellmann

## Agenda

- **Introduction**
- **Number Theory**
- **RSA**
- **Diffie-Hellmann**





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## Public-Key Technique 1

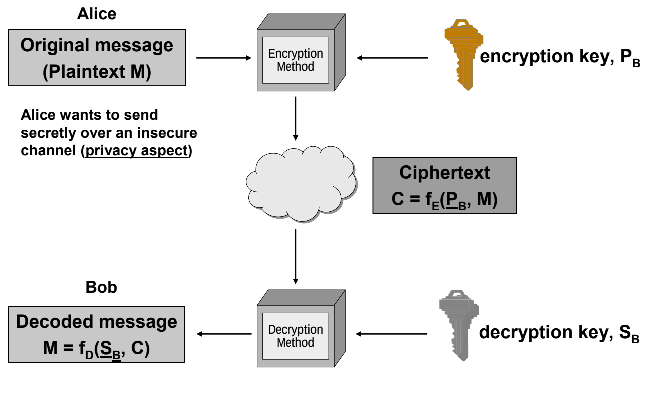
- **A pair of keys is used**
  - Private Key used by one party
    - key kept secret in one system
    - to sign messages to be sent to the other party for authentication
    - to decrypt messages received from the other party
  - Public Key used by the other party
    - key may widely be published to many systems
    - to encrypt messages to be sent to the other party for privacy
    - to verify messages received from the other party for authentication
- **Called asymmetric algorithms**



## Public-Key Technique 2

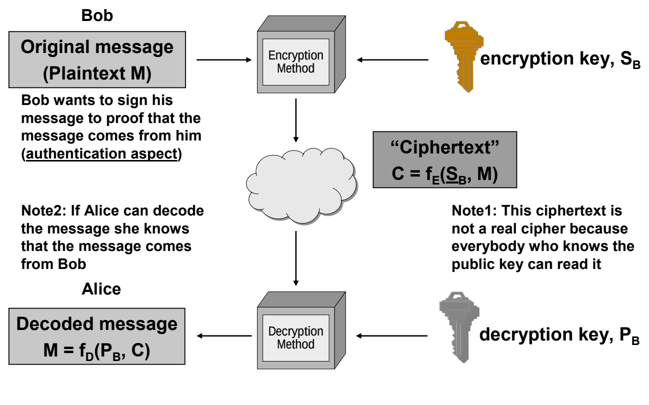
- **Methodology**
  - S … Private Key, P … Public Key
  - Security association between Alice and Bob
    - Alice generates one key pair ($P_A$, $S_A$), keeps $S_A$ secret in her system and give $P_A$ to Bob
  - Security association between Bob and Alice
    - Bob generates one key pair ($P_B$, $S_B$), keeps $S_B$ secret in his system and give $P_B$ to Alice
  - Encrypted messages from Alice to Bob
    - $C = f_E(\underline{P_B}, M)$
    - $M = f_D(\underline{S_B}, C)$ done by Bob to decrypt
  - Encrypted messages from Bob to Alice
    - $C = f_E(\underline{P_A}, M)$
    - $M = f_D(\underline{S_A}, C)$ done by Alice to decrypt

## Public-Key Technique (Privacy) 3






## Public-Key Technique (Authentication) 4







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## Public-Key Algorithms

- **RSA**
  - for encryption and digital signature
- **El-Gamal and DSS**
  - for digital signature
- **Diffie-Helman**
  - allows establishment of a shared secret
- **They are all very different from each other**
  - All hash algorithms do the same thing: they take a message and perform an irreversible transformation on it
  - All the secret-key algorithms do the same thing: they take a block and encrypt in a reversible way and allow message ciphers by chaining mechanism of blocks



## Agenda

- **Introduction**
- **Number Theory**
- **RSA**
- **Diffie-Hellmann**

## „Natürliche Zahlen“ 1

- **a teilt n -> b -> b * a = n**
- **b ist der komplementäre Teiler**
- **p = Primzahl („prime“)**
  - als Teiler nur 1 und p
  - Primzahlen sind ungerade Zahlen (Ausnahme p=2)
- **Untersuchung, ob eine Zahl Primzahl ist**
  - Überprüfen aller Zahlen bis Wurzel aus p
- **Primfaktorzerlegung („prime factorisation“)**
  - $240 = 24*10 = 3*8*2*5 = 2*2*2*2*3*5 = 2^4*3*5$
  - $3750 = 25*15*10 = 5*5*3*5*2*5 = 2*3*5^4$
  - Primfaktoren bleiben übrig



## „Natürliche Zahlen“ 2

- **ggT (größter gemeinsamer Teiler)**
  - gcd („greatest common divisor)
  - bilde Produkt der Primfaktoren, die in beiden Zerlegungen gleichzeitig vorkommen
  - $240 = 24*10 = 3*8*2*5 = 2*2*2*2*3*5 = \underline{2}^4*\underline{3}*\underline{5}$
  - $3750 = 25*15*10 = 5*5*3*5*2*5 = \underline{2}*\underline{3}*\underline{5}^4$
  - ggT (240, 3750) = 2*3*5 = 30
- **ggT (a, n) = 1 -> teilerfremd („relatively prime“)**
  - $54 = 2*3*3*3 = 2*3^3$
  - 65 = 5*13
  - ggT (54, 65) = 1









## „Natürliche Zahlen“ 3

- **kgV (kleinstes gemeinsames Vielfaches)**
  - bilde Produkt der höchsten Potenzen aller überhaupt vorkommender Primfaktoren
  - $75 = 3*5*5 = 3*\underline{5}^2$
  - $189 = 3*3*3*7 = \underline{7}*\underline{3}^3$
  - $kgV(75, 189) = 7*5^2*3^3 = 4725$
- **ggT (a, n) * kgV (a, n) = a * n**
  - $75 = 3*5*5 = \underline{3}*5^2$
  - $189 = 3*3*3*7 = 7*\underline{3}^3$
  - ggT (75, 189) = 3
  - $kgV(75, 189) = 7*5^2*3^3 = 4725$
  - 3 * 4725 = 14175 = 75 * 189



## ggT durch Euklid statt Faktorisierung

- **ggt lässt sich mit dem Euklid-Algorithmus leicht berechnen:**
  - Zwei Zahlen 792 (n) und 75 (a)
  - 792 = 10*75 + 42
    - (n = q*a + r) -> ggT(n,a) = ggT (a, r) mit 0 <= r < a
  - 75 = 1*42 + 33
  - 42 = 1*33 + 9
  - 33 = 3*9 + 6
  - 9 = 1*6 +3
  - 6 = 2* **3**
  - 3 ist ggT von 792 und 75

## Vielfachsummendarstellung des ggT

- **Vielfachsummendarstellung:**
  - ist g = ggT (n, a)
  - dann gibt es ganze Zahlen s und t so dass
  - g = t*a + s*n
  - "Lemma von Bezout"
  - die Werte t und s lassen sich mit dem **erweiterten Euklid-Algorithmus** leicht berechen



## Restklassenarithmetik – Modulo (mod)

- **Restklassenarithmetik verwendet natürliche Zahlen kleiner n**
  - 0, 1, 2, … n-1
- **Auf diese Zahlen lassen sich alle Operationen wie Addition, Multiplikation, Subtraktion anwenden**
- **Das Ergebnis einer solchen Operation wird durch n geteilt und der dabei entstandene Rest als modulo von n (mod n) bezeichnet**

  a mod n = r

  ist gleichbedeutend mit der Aussage

  es gibt eine ganze Zahl k gibt für die gilt: a = k * n + r





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## ggT und modulo

- **Wenn ggT (n, a) = 1 gilt laut Vielfachsummendarstellung**
  - 1 = t*a + s*n
- **Betrachtet man diese Gleichung bezüglich mod n**
  - 1 = t*a

    oder auch
  - (t*a) mod n = 1
    - t*a geteilt durch n ergibt Rest 1
    - Man schreibt auch anstelle von t ≡ $a^{-1}$ (mod n)
- **Dann ist a ist bezüglich modulo n invertierbar**
  - Es existiert daher ein komplementärer Teiler t der obige Gleichung erfüllt
  - Anmerkung:
    - in der normalen Arithmetik wäre t = 1/a = $a^{-1}$ (der Kehrwert von a) und daher ein Bruch (rationale Zahl)
    - In der Restklassenarithmetik gibt es aber keine Brüche sondern nur natürliche Zahlen
    - dennoch lässt sich hier (wenn a und n teilerfremd sind) mittels "**erweiterten Euklid**" eine natürliche Zahl t finden, die obige Gleichung erfüllt



## Introduction

- **Most of the public-key algorithms**
  - are based on modular arithmetic
- **Modular arithmetic**
  - uses non-negative integer numbers less than some positive integer ***n***
  - performs ordinary arithmetic operations like addition or multiplication on such numbers but replaces the ordinary arithmetic result ***a*** with its remainder ***r*** when divided by ***n***
  - the result ***r*** is expressed by "***a*** mod ***n***"
    - ***r* = *a*** mod ***n***
  - that means we can find an integer "***k***" such that
    - ***a* = *k* * *n* + *r***

## Modular Addition (mod 10)

K

| + | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 |
| 2 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 |
| 3 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 |
| 4 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 |
| 5 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 |
| 6 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 |
| 7 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 8 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 9 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

M C



## Encryption / Decryption (add mod 10)

- **Addition of constant K mod 10**
  - can be used for encryption of digits
  - each decimal digit maps to a different decimal digit
  - reversible way
  - constant K is the secret key
  - cipher like "Caesar Cipher"
- **Mono alphabetic substitution**
  - of course not a good cipher
- **Decryption**
  - subtracting the secret constant
  - if less than 0 then add 10









## Encryption / Decryption (add mod 10)

- **Decryption (cont.)**
  - instead of subtracting you can add the "additive inverse"
  - additive inverse of K is the number you have to add in order to get 0
    - e.g. 7 is the inverse of 3 because (3 + 7) mod 10 = 0
- **Encryption with K = 3**
  - (M + 3) mod 10 = C
- **Decryption with add-inv(K) = 7**
  - (C + 7) mod 10 = M

**Ciphertext**
**C = $f_E$(K, M)**

**Plaintext**
**M = $f_D$(K, C)**



## Modular Multiplication (mod 10)

K

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 0 | 2 | 4 | 6 | 8 | 0 | 2 | 4 | 6 | 8 |
| 3 | 0 | 3 | 6 | 9 | 2 | 5 | 8 | 1 | 4 | 7 |
| 4 | 0 | 4 | 8 | 2 | 6 | 0 | 4 | 8 | 2 | 6 |
| 5 | 0 | 5 | 0 | 5 | 0 | 5 | 0 | 5 | 0 | 5 |
| 6 | 0 | 6 | 2 | 8 | 4 | 0 | 6 | 2 | 8 | 4 |
| 7 | 0 | 7 | 4 | 1 | 8 | 5 | 2 | 9 | 6 | 3 |
| 8 | 0 | 8 | 6 | 4 | 2 | 0 | 8 | 6 | 4 | 2 |
| 9 | 0 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |

M C

## Encryption / Decryption (mult mod 10)

- **Encryption with K = 3**
  - (M * 3) mod 10 = C
- **Decryption**
  - multiplying C with the "multiplicative inverse"
  - multiplicative inverse of K is the number you have to multiply to get 1
    - (K * mult-inv(K)) mod n = 1
  - in ordinary arithmetic
    - mult-inv(K) = 1/K because K * 1/K = 1
    - 1/K is a fraction
  - but in modulo arithmetic we have only integers
    - so we cannot find mult-inv(K) for all possible values of K

**Ciphertext**
**C = $f_E$(K, M)**



## Decryption (mult mod 10)

- **Decryption (cont.)**
  - for our example we can find multiplicative inverse mod 10
    - just by trying if (K * mult-inv(K)) mod 10 = 1
  - so we find:
    - (1 * mult-inv(1)) mod 10 = 1 -> mult-inv(1) = 1
    - (3 * mult-inv(3)) mod 10 = 1 -> mult-inv(3) = 7
    - (7 * mult-inv(7)) mod 10 = 1 -> mult-inv(7) = 3
    - (9 * mult-inv(9)) mod 10 = 1 -> mult-inv(9) = 9
- **Therefore for K = 3 we can decrypt by multiplying with mult-inv(K) = 7**
  - (C * 7) mod 10 = M
- **Again that is not a good cipher**

**Plaintext**
**M = $f_D$(K, C)**





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## How to Find the Multiplicative Inverse

- **Finding multiplicative inverse t in mod n arithmetic**
  - is not obvious especially for a large number n
    - for a given a and n -> find number t such that (a*t) mod n = 1
  - brute-force search with n = 100 digit number will not work
- **But if the two numbers a and n are relatively prime**
  - then because of the general rule of "Bezout"
    - 1 = t*a + s*n
  - and finally if the rule of "Bezout" is performed modulo n
    - (t*a) mod n = 1
  - 1.) there must exists such inverse number t
  - 2.) the number t can be found by the Extended Euclid's algorithm
    - note to Euclid:
      - Basic Euclid's algorithm finds gcd (greatest common divisor) of two numbers (a, n)
    - alternatively t can be found by using the Euler formula $a^{\varphi(n)}$ mod n = 1
      - (t*a) mod n = $a^{\varphi(n)}$ mod n -> t = $a^{\varphi(n)-1}$ mod n
      - useful only if value for φ(n) is known



## Relatively Prime

- **Why are the numbers 1, 3, 7 and 9**
  - the only ones with multiplicative inverse mod 10?
- **They are the only ones which are relatively prime to 10**
  - each of these numbers does not share any common factors with 10 other than 1 (gcd is 1)
    - the largest integer that divides 9 and 10 is 1
    - the largest integer that divides 7 and 10 is 1
    - the largest integer that divides 3 and 10 is 1
    - but 6 and 10 have two factors in common -> 1 and 2

## Euler Function

- **In turns out that in general**
  - when working with mod n all numbers which are relatively prime to n have a multiplicative inverse but none of the other numbers
- **How many numbers less than n are relatively prime to n?**
  - notation for that amount is **φ(n)**
  - **φ(n) …. Euler function**
  - there is no simple formula but in special cases of n we can determine φ(n)
    - -> see next slides



## φ(n) for Prime Numbers

- **If n = prime then φ(n) = n-1**
  - all the integers 1, 2, … n-1 are relatively prime to n
    - greatest common divisor with n is 1 for all of them
  - note:
    - a number n is prime means that the only divisors are 1 and n itself
    - it cannot written as a product of other numbers
- **If n = product of two distinct prime numbers p and q then**

  **φ(n) = (p-1)*(q-1)**





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## Explanation φ(n)

- **Why?**
  - p and q are prime numbers and n = p*q
    - n could be divided only by p or q
    - n could not divided by other numbers because p and q are prime (only 1, p or q are possible as factors)
  - there are n = (p*q-1) numbers -> (1,2, … n-1)
    - we want to exclude all numbers which are not relatively prime
    - those are the numbers that are either multiples of p or q
      - we have the numbers 1q, 2q, … (p-1)q which are in sum p-1 multiples of q which are less than p*q
      - we have the numbers 1p, 2p, … (q-1)p which are in sum q-1 multiples of p which are less than p*q
    - the rest must be relatively prime
  - φ(n) = (p*q -1) - (p-1) - (q-1) = p*q - 1 - p + 1 - q + 1 =

    p*q - p - q + 1 = (p - 1) * (q - 1)



## Example

- **p = 5, q = 7**
- **p*q = 35 = n**
- **1*7, 2*7, 3*7, 4*7**
  - are not relatively prime to 35 because greatest common divisor is 7 instead of 1
  - that is p-1 times -> 4 times
- **1*5, 2*5, 3*5, 4*5, 5*5, 6*5**
  - are not relatively prime to 35 because greatest common divisor is 5 instead of 1
  - that is q-1 times -> 6 times
- **φ(35) = (p-1)*(q-1) = 24**
  - all other numbers are relatively prime to 35

## Example

**p = 5, q = 7**

**p*q = 35 = n**

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 15 | 16 | 17 | 18 | 19 | 20 | 21 |
| 22 | 23 | 24 | 25 | 26 | 27 | 28 |
| 29 | 30 | 31 | 32 | 33 | 34 | 35 |



## Euler

- **Euler's theorem**

  if two numbers a and n are relatively prime then

  $a^{\varphi(n)}$ mod n = 1 (Euler 1.)

  and for any integer k

  $a^{1+k*\varphi(n)}$ mod n = a (Euler 2.)

  that is because

  $(a^1 * a^{k*(\varphi(n))})$ mod n = $(a^1 * a^{(\varphi(n))k})$ mod n = a * $1^k$ = a

- **It turns out that**

  if n = product of two distinct prime numbers p and q then the formula's of Euler are valid even if the number a is not relatively prime to n (so for all x with x <= n)

  φ(n) = (p-1)*(q-1)

  $x^{(p-1)*(q-1)}$ mod n = 1 (Euler 3.)

  $x^{1+k*(p-1)*(q-1)}$ mod n = x (Euler 4.)





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## Modular Exponentiation (mod 10)

K

| $X^Y$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 4 | 8 | 6 | 2 | 4 | 8 | 6 | 2 | 4 | 8 | 6 |
| 3 | 1 | 3 | 9 | 7 | 1 | 3 | 9 | 7 | 1 | 3 | 9 | 7 | 1 |
| 4 | 1 | 4 | 6 | 4 | 6 | 4 | 6 | 4 | 6 | 4 | 6 | 4 | 6 |
| 5 | 1 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 6 | 1 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 7 | 1 | 7 | 9 | 3 | 1 | 7 | 9 | 3 | 1 | 7 | 9 | 3 | 1 |
| 8 | 1 | 8 | 4 | 2 | 6 | 8 | 4 | 2 | 6 | 8 | 4 | 2 | 6 |
| 9 | 1 | 9 | 1 | 9 | 1 | 9 | 1 | 9 | 1 | 9 | 1 | 9 | 1 |

M C



## Encryption (exp mod 10)

**Ciphertext**
**C = $f_E$(K, M)**

- **Encryption with K = 3**
  - $(M^3)$ mod 10 = C
  - K = 3 or K = 7 can be used
    - because all input digits are mapped to a unique output digit
  - K = 2 or 4 or 6 or 8 cannot be used
    - because some input digits are mapped to the same output digit
  - K = 1 or K = 5 or K = 9 makes no sense
    - because input and output digits are the same
- **Exists exp-inverse of K for decryption?**
  - exp-inverse is the number so that $(M^K)^{exp\text{-}inv(K)}$ mod 10 = M
    - note: in ordinary arithmetic exp-inv(K) = 1/K = $K^{-1}$
  - Yes sometimes!

## Decryption Aspects (exp mod 10)

- **Example mod 10:**
  - only numbers 1, 3, 7 and 9 are relative prime to 10
  - $\varphi(10) = 4$
  - for K = 3

    $K^{\varphi(n)} \bmod n = 1$ is valid because of Euler
    Proof : $3^4 \bmod 10 = 81 \bmod 10 = 1$
    $(t*K) \bmod n = 1$ because of lemma of bezout performed modulo n
  - exp-inv(K) can be found by using

    $(t*K) \bmod n = 1 = K^{\varphi(n)} \bmod n$ -> $t = K^{\varphi(n)-1} \bmod n$

    (exp-inv(3)*3) mod 10 = $3^4$ mod 10 -> exp-inv(3)* = $3^{4-1}$ mod 10 = 27 mod 10 = 7
- **Therefore for K = 3 decryption with exp-inv(K) = 7**
  - $(C^7) \bmod 10 = M$
  - examples
    - $2^3 \bmod 10 = 8$ , $8^7 \bmod 10 = 2097152 \bmod 10 = 2$
    - $3^3 \bmod 10 = 7$ , $7^7 \bmod 10 = 823543 \bmod 10 = 3$
    - $4^3 \bmod 10 = 4$ , $4^7 \bmod 10 = 16384 \bmod 10 = 4$

**Plaintext**
**M = $f_D$(<u>K</u>, M)**



## Important Rule for Exponentiation

- **Looking to the table**
  - columns 1 and 5 are the same, 2 and 6 are the same, 3 and 7 are the same, 4 and 8 are the same, …
- **Theory proofs**
  - $x^y \bmod n = x^{y \bmod \varphi(n)} \bmod n$
    - this formula is not true for all values of n but it is true if n is prime or the product of two primes for any x with x < n
  - in our case n = 10 (prime factors are p=2 , q=5)
    - numbers 1, 3, 7 and 9 are relative prime to 10
    - $\varphi(10) = 4 = (p-1)*(q-1) = 1 * 4$
    - $x^y \bmod 10 = x^{(y \bmod 4)} \bmod 10$
    - so column i+4th and ith are the same
- **Special important case for public-key algorithm (RSA)**
  - if <u>y mod φ(n) = 1</u> then for any number x

    $x^y \bmod n = x \bmod n$ or $(x^y = x) \bmod n$
  - we need this trick for later for to perform RSA decryption
    - modulo function is a one-way function, in order to reverse we need a trapdoor, this trick shows the possible trapdoor





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## Agenda

- **Introduction**
- **Number Theory**
- **<u>RSA</u>**
- **Diffie-Hellmann**



## RSA algorithm

- **generate public and private keys**
  - two large primes: p and q
  - build n = p*q
  - $\varphi(n) = (p-1)*(q-1)$
  - p and q remains secret
  - select **<u>e</u>** that is relatively prime to $\varphi(n) = (p-1)*(q-1)$
    - use Basic Euclid to proof you have found such an e
      - $\gcd(\varphi(n), e) = 1$ with $1 < e < \varphi(n)$
    - therefore it must also exists a multiplicative inverse d of e
      - d such that $(e*d) \bmod \varphi(n) = 1$
    - note: because of (Euler 3.)

      $e^{\varphi(n)} \bmod n = 1$ is valid

      from (Euler 1.): $a^{\varphi(n)} \bmod n = 1$ is valid if a and n are relatively prime to each other but in our case e is only relatively prime to $\varphi(n)$;

      but if n = p*q (product of two primes) then (Euler 1.) is valid for all numbers x with x less n (Euler 3.) -> hence also for number e

## RSA algorithm

- **generate public and private keys (cont.)**
  - find **d** that is the multiplicative inverse of e
    - that is (e*d) mod φ(n) = 1
  - d can be found using the **Extended Euclid** algorithm
- **now the public key is <e, n>**
- **and the private key is <d, n>**
- **it is not feasible**
  - to determine the private key from the public key
    - you need to know p and q in order to build φ(n)
  - but factoring is a hard problem
    - finding p and q based on n



## RSA Encryption / Decryption

- **encryption with public key**
  - divide the plaintext message (regarded as bit string) into blocks of M's where every M falls in the interval $0 < M < n$
  - compute $C = M^e \bmod n$ (with public key)
  - decryption $M = C^d \bmod n$ (with private key)
  - privacy aspect
- **sign a message using private key**
  - compute $S = M^d \bmod n$ (with private key)
  - verification $M = S^e \bmod n$ (with public key)
  - authentication and integrity aspect







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## RSA Algorithm Proof

- **Proof of correctness**
  - we have chosen (e*d) mod φ(n) = 1
    - that means that we can write e*d = 1 + k*φ(n) = 1 + k*(p-1)*(q-1)
  - because of (Euler 4.) the following rule is valid
    $x^{1+k*(p-1)*(q-1)} \bmod n = x$ for all x less then n
  - therefore for any x if we build $x^{ed} \bmod n$
    - then $x^{ed} \bmod n = x^{(1 + k*(p-1)*(q-1))} \bmod n = x \bmod n$
  - if we encrypt and decrypt:
    - Encrypt: $C = M^e \bmod n$
    - Decrypt: $M = C^d \bmod n$

      $\underline{C^d = (M^e)^d = (M^{ed}) = M \bmod n}$
  - If we sign and verify:
    - Sign: $S = M^d \bmod n$
    - Verify: $M = S^e \bmod n$

      $\underline{S^e = (M^d)^e = M^{de} = M \bmod}$ n
  - q.e.d



## RSA Example

- **p = 3, q = 11**
- **hence n = 33, φ(n) = 20**
- **choose e = 3 relatively prime to 20**
  - take 3 (gcd = 1)
- **compute d = 7**
  - 3d = 1 (mod 20)
- **encryption $C = M^3 \bmod 33$**
- **decryption $M = C^7 \bmod 33$**
- **M < 33**
  - therefore encode every letter of the message as single block; numbers 1 … 26 represent A … Z

## RSA Example (cont.)

| Plaintext Sender | | $M^3$ | $M^3$ (mod 33) | $C^7$ | $C^7$ (mod 33) | |
|---|---|---|---|---|---|---|
| S | 19 | 6859 | 28 | 13492928512 | 19 | S |
| U | 21 | 9621 | 21 | 1801088541 | 21 | U |
| Z | 26 | 17576 | 20 | 1280000000 | 26 | Z |
| A | 01 | 1 | 1 | 1 | 01 | A |
| N | 14 | 2744 | 5 | 78125 | 14 | N |
| N | 14 | 2744 | 5 | 78125 | 14 | N |
| E | 05 | 125 | 26 | 8031810176 | 05 | E |

Ciphertext

Plaintext Receiver



## RSA Security 1

- **Security depends on difficulty of factoring**
  - find p and q from n
- **If you can factor quickly you can break RSA**
  - factoring the public value n into p and q, building φ(n) and knowing public value e allows you to find d by performing the same computations (Euclid's algorithm) as done by the key generation
- **Fortunately factoring is a hard problem**
  - Currently 500-bit numbers are the largest which can be factorized
- **The minimum length proposed for n is 1024 bits and for p and q 512 bits each**









## RSA Security 2

- **There is a possible misuse**
  - an eavesdropper can guess what is going to be transmitted although he is not able to decrypt
  - he can encrypt all expected messages with the public key and compares the result with the ciphertext he has eavesdropped
    - especially a problem with short messages
  - if there is a match then he knows what was transmitted
- **Therefore is necessary to use special guidelines of how to format RSA message**
  - e.g. short messages should be concatenated with a large random number (e.g. 64 bits)
  - PKCS … Public-Key Cryptography Standard



## RSA Performance Aspects 1

- **Finding large prime numbers**
  - typically greater than $100^{100}$
  - pseudo prime numbers (random numbers) are used and checked with Fermat´s theorem
    - If n is prime then for 0 < a < n -> $a^{n-1}$ mod n = 1
      - taken from (Euler 1.): $a^{\varphi(n)}$ mod n = 1 if a and n are relatively prime
      - if n is prime then φ(n) =n-1
      - If n is really a prime number all numbers a are relatively prime and Fermat's rule is valid
  - primality test:
    - For a number n to check pick a number a < n
    - If $a^{n-1}$ mod n = 1 is not fulfilled then n is certainly not prime
    - If $a^{n-1}$ mod n = 1 is fulfilled then n may or may not be prime (risk for failure is 1 to $10^{13}$ in case of randomly generated number of about hundred digits)
    - Try multiple values of number a to make the test more reliable
    - Special attention for Carmichael numbers

## RSA Performance Aspects 2

- **Finding e and d**
  - Euclid´s algorithm
    - normal -> gcd of e and φ(n) = 1
    - extended -> multiplicative-inverse
- **Exponentiation with big numbers**
  - square and multiply algorithm and do modular reduction after each multiply
    - e.g. result of $123^{32}$ mod 678
      - $123^2$ = 123*123 = 15129 = 213 mod 678
      - $123^4$ = 213*213 = 45369 = 621 mod 678
      - $123^8$ = 621*621 = 385641= 537 mod 678
      - $123^{16}$ = 537*537 = 288369 = 219 mod 678
      - $123^{32}$ = 219*219 = 47961 = 501 mod 678



## RSA Facts

- **RSA is special type of block cipher**
- **Variable key-length**
  - usually 512 - 2048 bits
  - compromise between enhanced security and efficiency
  - plaintext block need to be smaller than the key length
- **Ciphertext block will be the length of the key**
- **Typically much slower to implement than conventional block ciphers like DES or IDEA**
  - unsuitable for encrypting large messages
  - 1000 times (HW) to 100 times (SW) slower
  - mostly used to encrypt a session key for performing a secret-key algorithm







## Agenda

- **Introduction**
- **Number Theory**
- **RSA**
- **Diffie-Hellmann**



## Diffie-Hellman (DH)

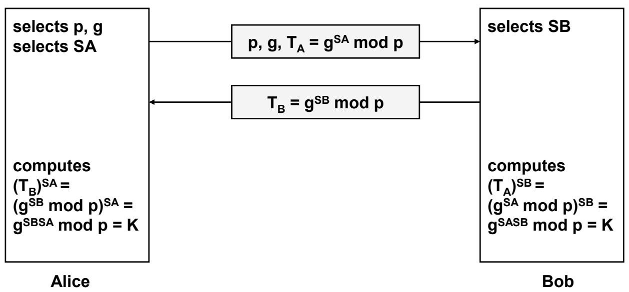
- **Oldest public-key system still in use**
- **Compared to RSA**
  - it does neither encryption nor signatures
- **Allows two individuals to agree on a shared key**
  - even though they can only exchange information in public
  - this secret is used for symmetric encryption
  - better performance than doing this with RSA
- **DH used for key establishment**

## Diffie-Hellman Algorithm 1

- **Select p prime and g < p**
  - g and p might be public, strong primes are used for p
    - -> more secure if (p-1)/2 is also a prime
- **Select random secret numbers SA, SB**
  - SA,SB are the private DH keys
  - Alice: SA -> $T_A = g^{SA}$ mod p
  - Bob: SB -> $T_B = g^{SB}$ mod p
- **Exchange $T_A$ and $T_B$**
  - these are the public DH keys
- **Produce shared key K**
  - Alice: $K = T_B^{SA} = (g^{SB})^{SA} = g^{SBSA} = g^{SASB}$ mod p
  - Bob: $K = T_A^{SB} = (g^{SA})^{SB} = g^{SASB} = g^{SBSA}$ mod p



## Diffie-Hellman Algorithm 2









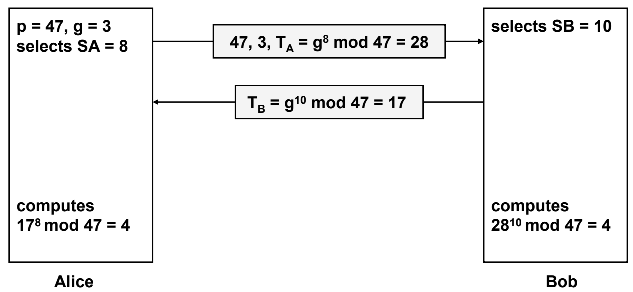



## Diffie-Hellman Algorithm 3

- **Nobody else can compute**
  - SA or SB and hence $g^{SASB}$ in reasonable time
  - even though he knows $T_A = g^{SA}$ mod p
  - even though he knows $T_B = g^{SB}$ mod p
- **Based on the fact**
  - that discrete logarithm modulo a very large prime number is hard to compute
  - no one has found any efficient method to do this



## Diffie-Hellman Example

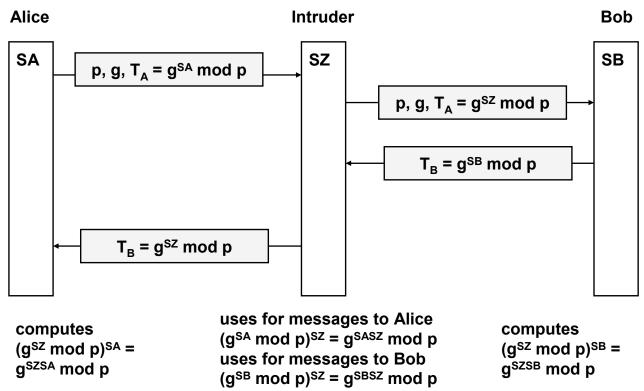


## Man-in-the-middle Attack 1

- **Problem:**
  - when $T_A$ and $T_B$ are sent over a public network and an active intruder is present
  - Alice will establish a secret key with whoever transmitted $T_B$ but it might not be Bob
  - Bob will establish a secret key with whoever transmitted $T_A$ but it might not be Alice
- **There is no authentication (!) in the key exchange process of DH**
  - vulnerable against man-in-the-middle attack
  - also known as bucket brigade attack
- **DH is only secure against passive attacks**



## Man-in-the-middle Attack 2







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## Man-in-the-middle Attack 3

- **Man-in-the-middle Attack**
  - Intruder intercepts communication, impersonates in both directions
  - intruder handles a different secret key with every party
- **Solutions to overcome this attack**
  - published public numbers
    - e.g. in the newspapers or PKI techniques
    - not possible to make active attacks, because the advertised values cannot be changed or cannot at least be easily changed
  - authentication via out-band channel
    - $T_A$ and $T_B$ are transmitted by voice communication over telephone network
    - authentication based on voice recognition of other party



## Man-in-the-middle Attack 3

- **Solutions to overcome this attack (cont.)**
  - authentication via encryption
    - Alice and Bob know some sort of secret e.g. each other's public-key
    - encrypt DH value with other side's public-key
    - receiver can decrypt the message by using its own corresponding private-key